Journal of Approximation Theory **105**, 129–132 (2000)
doi:10.1006/jath.2000.3462, available online at http://www.idealibrary.com on IDEAL®

Common Zeros of Two Polynomials in an Orthogonal Sequence

Peter C. Gibson

Department of Mathematics & Statistics, University of Calgary, Calgary, Canada T2N 1N4

Communicated by Alphonse Magnus

Received September 28, 1998; accepted in revised form January 31, 2000

We show that for any positive integers $k<m$ there exists a sequence $p_0, ..., p_m$ of orthogonal polynomials (p_i having degree i) such that p_k and p_m have $\min\{k, m-k-1\}$ zeros in common, the maximum possible. More generally, if, in a sequence $p_0, ..., p_m$ of orthogonal polynomials, p_k and p_m have no common zero, then for every n $(m+1 \leqslant n \leqslant m+k)$, there exists an orthogonal sequence $q_0, ..., q_n$ such that (i) $q_k = p_k$ and (ii) the zeros of q_n are precisely the zeros of p_m together with $n-m$ zeros of p_k.

Key Words: orthogonal polynomials; common zeros.

1. INTRODUCTION

Let p_0, p_1, p_2, ... be a sequence of orthogonal polynomials, where the degree of p_i is i. Fix positive integers $k<n$, and let $z_1 < \cdots < z_k$ denote the zeros of p_k. A classical interlacing theorem (see [2, Theorem 3.3.3], for instance) states that each of the $k+1$ open intervals

$$(-\infty, z_1), (z_1, z_2), ..., (z_{k-1}, z_k), (z_k, \infty)$$

contains at least one zero of p_n. This establishes that at least $k+1$ zeros of p_n are distinct from the zeros of p_k, or, equivalently, $p_n(z_i)=0$ for at most $n-k-1$ values of i. Thus we have a general bound:

$$p_k \text{ and } p_n \text{ have at most } \min\{k, n-k-1\} \text{ zeros in common.} \tag{1}$$

From the theoretical point of view, it is natural to ask whether there exist other general restrictions on the number of common zeros between two polynomials in an orthogonal sequence, apart from the interlacing property just cited or whether the bound (1) is sharp. In fact, the bound (1) *is* sharp in the following sense.

0021-9045/00 $35.00

THEOREM. *For any positive integers $k<n$, there exists a sequence $q_0, ..., q_n$ of monic, orthogonal polynomials such that q_k and q_n have precisely $\min\{k, n-k-1\}$ zeros in common.*

We prove this, as well as a more general result, Theorem 1, in Section 2. The subsection below is devoted to terminology and notation.

1.1. *Terminology and Notation*

A sequence $p_0, p_1, p_2, ...$ of real polynomials in one variable is *orthogonal* if, for some measure μ on the real line $\mathbf{R}$,

$$0<\int_{\mathbf{R}} p_i^2\, d\mu<\infty \qquad \text{and} \qquad \int_{\mathbf{R}} p_i p_j\, d\mu=0, \tag{2}$$

for every $i, j \geqslant 0$ where $i \neq j$.

In the present article, the index i of the polynomial p_i in such a sequence will always coincide with the degree of p_i.

We confine our attention to finite sequences of monic polynomials. This does not result in any loss of generality. Note that if

$$p_0, p_1, ..., p_n$$

is a finite sequence of monic, orthogonal polynomials, then there exists a measure μ, satisfying (2), which is supported on a finite set. Also, $p_0 \equiv 1$.

A *Jacobi* matrix is a symmetrical, tridiagonal matrix whose next-to-diagonal elements are strictly positive.

The symbol P_n denotes the space of polynomials having degree $\leqslant n$.

2. THE MAIN RESULT

THEOREM 1. *Let $k<m<n$ be positive integers where $n-m \leqslant k$. Suppose that $p_0, ..., p_m$ is a sequence of monic, orthogonal polynomials such that p_k and p_m have no common zero. Let $z_1, ..., z_{n-m}$ be any $n-m$ zeros of p_k. Then there exists a sequence $q_0, ..., q_n$ of monic, orthogonal polynomials satisfying*:

(i) $q_k = p_k$;

(ii) *the zeros of q_n are precisely the zeros of p_m together with $z_1, ..., z_{n-m}$.*

Proof. Let μ be a measure with respect to which $p_0, ..., p_m$ are orthogonal. Let $\lambda_1 < \cdots < \lambda_m$ be the zeros of p_m, and let $w_1, ..., w_m$ be the corresponding weights for the m-point Gauss–Jacobi quadrature formula associated to $d\mu$ (see [2, Sect. 3.4]). Set

$$\tilde{\lambda}_i = \begin{cases} \lambda_i & (1 \leqslant i \leqslant m) \\ z_{i-m} & (m+1 \leqslant i \leqslant n) \end{cases}.$$

In addition, set $\tilde{w}_i = w_i$ for $1 \leqslant i \leqslant m$. Choose $\tilde{w}_{m+1}, ..., \tilde{w}_n$ to be arbitrary, positive real numbers. The hypothesis that p_k and p_m have no common zeros implies that the $\tilde{\lambda}_i$ are all distinct. The discrete scalar product $\langle \cdot, \cdot \rangle$ on $\mathbf{P}_{n-1}$ defined by

$$\langle p, q \rangle = \sum_{i=1}^{n} p(\tilde{\lambda}_i)\, q(\tilde{\lambda}_i)\, \tilde{w}_i$$

engenders a sequence $q_0, ..., q_n$ of monic, orthogonal polynomials, where each q_k $(1 \leqslant k \leqslant n)$ is uniquely characterized as being monic of degree k and orthogonal to $\mathbf{P}_{k-1}$ with respect to $\langle \cdot, \cdot \rangle$. (See [1].) We claim that p_k is orthogonal to $\mathbf{P}_{k-1}$ with respect to $\langle \cdot, \cdot \rangle$. To see this, let $r \in \mathbf{P}_{k-1}$. Then

$$\begin{aligned} \langle r, p_k \rangle &= \sum_{i=1}^{n} r(\tilde{\lambda}_i)\, p_k(\tilde{\lambda}_i)\, \tilde{w}_i \\ &= \sum_{i=1}^{m} r(\lambda_i)\, p_k(\lambda_i)\, w_i \qquad \text{(by choice of the } \tilde{\lambda}_i \text{ and } \tilde{w}_i\text{)} \\ &= \int_{\mathbf{R}} r p_k \, d\mu \qquad \text{(exactness of Gauss–Jacobi quadrature on } \mathbf{P}_{2m-1}\text{)} \\ &= 0 \qquad \text{(since } p_k \perp \mathbf{P}_{k-1} \text{ with respect to } d\mu\text{).} \end{aligned}$$

It follows that $p_k = q_k$, since p_k is monic of degree k.

To see that (ii) holds, note that the zeros $\lambda_1, ..., \lambda_m$ of p_m are among the zeros $\tilde{\lambda}_1, ..., \tilde{\lambda}_n$ of q_n. And $z_1, ..., z_{n-m}$ are also among the zeros $\tilde{\lambda}_1, ..., \tilde{\lambda}_n$ of q_n. ∎

The hypothesis of Theorem 1 elicits the question: Does there exist, for all positive integers $k < m$, a sequence $p_0, ..., p_m$ with the property that p_k and p_m have no common zero? In fact almost every m-element sequence has this property. Before sketching why this is true, we recall the well-known connection between orthogonal polynomials and Jacobi matrices. A sequence $p_0, ..., p_m$ of monic, orthogonal polynomials satisfies a recursion of the form

$$\begin{aligned} p_1(x) &= x - a_1 \\ p_k(x) &= (x - a_k)\, p_{k-1}(x) - b_{k-1} p_{k-2}(x) \qquad (2 \leqslant k \leqslant m), \end{aligned} \tag{3}$$

where $b_i > 0$ $(1 \leqslant i \leqslant m-1)$. The associated Jacobi matrix

$$J = \begin{pmatrix} a_1 & \sqrt{b_1} & 0 & \cdots & 0 \\ \sqrt{b_1} & a_2 & \sqrt{b_2} & & \vdots \\ 0 & \sqrt{b_2} & \ddots & \ddots & 0 \\ \vdots & & \ddots & & \sqrt{b_{m-1}} \\ 0 & \cdots & 0 & \sqrt{b_{m-1}} & a_m \end{pmatrix}$$

has the property that p_i is the characteristic polynomial of the ith leading, principal submatrix J_i of J for $1 \leqslant i \leqslant m$. In this way, there is a bijective correspondence between the class of $(m+1)$-element sequences $p_0, ..., p_m$ of monic, orthogonal polynomials and the class of $m \times m$ Jacobi matrices. (See [1] for details.) Now, fix J as above and let $\tilde{J}$ denote the matrix obtained from J by replacing the (m, m) entry, a_m, by a parameter $\tilde{a}_m$. The matrix $\tilde{J}$ gives rise to the same initial sequence $p_0, ..., p_{m-1}$ as J. But (by consideration of the recursion (3)), the characteristic polynomial $\tilde{p}_m$ of $\tilde{J}$ and p_m have a common zero only if $\tilde{a}_m = a_m$, in which case $\tilde{p}_m = p_m$. Now, suppose that, for $\tilde{a}_m = t$, $\tilde{p}_m$ and p_k do have a common zero. Then, for every sufficiently small perturbation $\tilde{a}_m = t + \varepsilon$, the corresponding $\tilde{p}_m$ has no zero in common with p_k. Therefore the values of $\tilde{a}_m$ for which $\tilde{p}_m$ and p_k have a common zero are isolated and hence have measure zero.

THEOREM 2. *For any positive integers $k < n$, there exists a sequence $q_0, ..., q_n$ of orthogonal polynomials such that q_k and q_n have precisely* $\min\{k, n-k-1\}$ *zeros in common.*

Proof. To avoid the trivial case $\min\{k, n-k-1\} = 0$, suppose $0 < k < n-1$. If $\min\{k, n-k-1\} = k$, apply Theorem 1 with $m = n-k$. Otherwise apply the theorem with $m = k+1$. ∎

REFERENCES

1. C. de Boor and G. H. Golub, The numerically stable reconstruction of a Jacobi matrix from spectral data, *Linear Algebra Appl.* **21** (1978), 245–260.
2. G. Szegö, "Orthogonal Polynomials," revised ed., American Mathematical Society, New York, 1959.